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7

Goal Formation, Planning, and Learning by Pre-School Problem Solvers or: "My Socks are in the Dryer"

David Klahr
Carnegie-Mellon University

INTRODUCTION

This is a report on a project aimed at understanding and ultimately improving the problem-solving abilities of young children. There are both practical and theoretical grounds for such an effort. The practical justification comes from the observation that problem-solving abilities are implicitly assumed in early school activities: for example, in the puzzles and games used to teach and test fundamentals of reading and arithmetic. Not only are rudimentary problem-solving skills assumed in the early curricula, but also advanced and general problem-solving skills are an explicit goal of subsequent instruction. We have all heard the claim that training in mathematics or reading (or your favorite subject) enhances the ability of students to think logically. It is somewhat curious, then, that for all the implicit and explicit emphasis on problem-solving skills, they are rarely taught directly to young children. One long-range goal of this project, then, is to instruct preschool children on general problem-solving methods.

The theoretical interest in such a study derives from a look at the other side of the coin: Children learn about problem solving, even without direct instruction. Both intuition and casual observation indicate that, as children approach school age, they acquire a range of problem-solving abilities that are typically characterized simply as "common sense."

Let me give an example of what I mean by common sense reasoning in a young child. Consider the following scenario:

Scene: Child and father in yard. Child's playmate appears on bike.
Child: Daddy, would you unlock the basement door?
Daddy: Why?
Child: 'Cause I want to ride my bike.

Daddy: Your bike is in the garage.
 Child: But my socks are in the dryer.

What kind of weird child is this? What could possibly explain such an exchange? Let me propose a hypothetical sequence of the child's mental activity:

Top goal: ride bike.
 Constraint: shoes or sneakers on.
 Fact: feet are bare.
 Subgoal 1: get shod.
 Fact: sneakers in yard.
 Fact: sneakers hurt on bare feet.
 Subgoal 2: protect feet (get socks).
 Fact: sock drawer was empty this morning.
 Inference: socks still in dryer.
 Subgoal 3: get to dryer.
 Fact: dryer in basement.
 Subgoal 4: enter basement.
 Fact: long route through house, short route through yard entrance.
 Fact: yard entrance always locked.
 Subgoal 5: unlock yard entrance.
 Fact: Daddies have all the keys to everything.
 Subgoal 6: ask daddy.

The *example* is real (in fact it is from my own experience) and should be plausible to everyone who has spent time around young children. On the other hand, the *analysis* of the example is less convincing, based as it is on a host of assumptions. Some of these assumptions are easily testable. We could determine whether the child knows *constraints*, such as the one about riding bikes only when shod. Similarly, we could assess the child's knowledge of *facts* about dryer location, shortest route to the basement, and so on. Somewhat more difficult, but still reasonable, would be the job of finding out what sorts of *inferences* the child was capable of making about her day-to-day environment, such as the one about where the socks might be, given that they were not in the drawer. However, the dominant feature of the hypothesized thought sequence is not any one of these features in isolation. Rather, it is their organization into a systematic means-ends chain. Thus, I am suggesting that by the time the child is old enough to exhibit the sort of behavior just described, she has already acquired some *general problem-solving* processes. These enable her to function effectively — that is, to achieve desired goals — by noticing relevant features of the environment and organizing a wide range of facts, constraints, and simple inferences in some systematic manner.

As it stands, such a suggestion is unremarkable. The interesting questions concern the detailed nature of such processes, their generality, and their developmental course. Paraphrasing Newell & Simon (1972, p. 663), the question is

whether we can view the child in some task environments as an information-processing system, and if so, whether we can identify problem spaces, search strategies, heuristics, goal structures, and so on in a relatively precise fashion. Furthermore, can we determine which aspects of problem solving derive from the task environment and which from characteristics of the subject?

There are two rather distinct approaches one could take in studying children's problem solving. One approach, suggested by Charlesworth (1976), would study the child from an ethological perspective and observe the occurrences of everyday problem solving in the child's normal environment. This is clearly a laborious and time-consuming way to go about the task, although the naturalistic approach has certainly enriched our knowledge about early language development, and it may be the only way to observe interesting problem-solving episodes in children less than 2 years old. The other approach, which we have taken, is to study in the laboratory formal problems whose structure we can thoroughly analyze and over whose systematic variants we can maintain reasonable control. Although Neisser (1976) inveighs against such "artificial" and "academic" environments, it seems unlikely that young children acquire and hold in reserve a special set of cognitive processes for laboratory experiments that bear little relation to those they use in the "natural" environment of rooms, houses, toys, cars, and playgrounds.

Thus, the initial phase of this project has focused on the performance of children between the ages of 3 to 5 years on a variety of well-defined tasks. The first task in the series, and the only one I will report on here, is the Tower of Hanoi.

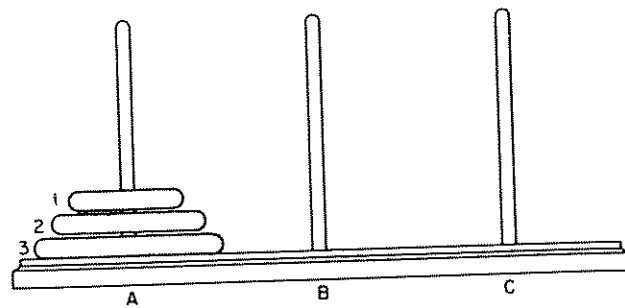
THE TOWER OF HANOI

The standard version of this task consists of a series of three pegs and a set of n disks of decreasing size.¹ The disks sit initially on one of the pegs, and the goal is to move the entire n -disk configuration to another peg, subject to two constraints: Only one disk can be moved at a time, and at no point can a larger disk be above a smaller disk on any given peg. A standard three-disk problem is shown in Fig. 7.1.

To solve this problem you might reason as follows:

I have to build the stack up from the bottom, which means that I must get disk 3 from A to C, but 2 is in the way, so I'll have to move 2 to B. But

¹The many uses of this task reveal some of the changing goals and methods of experimental psychology over the last 50 years (cf. Byrnes & Spitz, 1977; Cook, 1937; Egan & Greeno, 1974; Gagné & Smith, 1962; Hormann, 1965; Klix, 1971; Neves, 1977; Peterson, 1929; Piaget, 1976; Simon, 1975; Sydow, 1970).



Problem: Move all the disks from peg A to peg C.

FIG. 7.1 Three-disk Tower of Hanoi problem.

if I want to move 2 to B, I must first get 1 out of the way, so my first move will be 1 to C. Now let me reconsider the new configuration. To get 3 to C, I still have to move 2 to B, which I can now do. Now to get 3 to C I must remove 1 from C, so I will put it on B, and at last I can move 3 to C. And so on.

Although, as will become evident, there are other ways to solve the problem, the example shows that even this simple version of the puzzle can tax one's ability to coordinate sequential reasoning, perceptual discrimination, quantitative ordering, and short-term memory processes. The task involves a well-defined initial state, an unambiguous desired state, and a very limited set of rules about how to change states. The difficulty lies in organizing a sequence of rule applications (legal moves) that ultimately transform the initial physical configuration into the desired one.

Children's Version of the Puzzle: Monkey Cans

For use with young children, we modified the task in three ways that changed its superficial appearance while maintaining its basic structure.

Materials. We use a set of nested inverted cans as shown in Fig. 7.2. The cans are modified so that they fit very loosely on the pegs; when they are stacked up, it is impossible to put a smaller can on top of a larger can. Even if the child forgets the relative size constraint, the materials provide an obvious physical consequence of attempted violations: Little cans simply fall off of bigger cans. Furthermore, the materials are intuitively more "reasonable" in two regards. First, unlike the standard problem in which small disks may obstruct larger ones, with these materials, bigger cans obstruct smaller cans, either by sitting atop them or by being on a goal peg. Second, larger cans not only sit on top of but also partially contain the smaller cans. Each can is a different color and makes a satisfying clunk with each move.

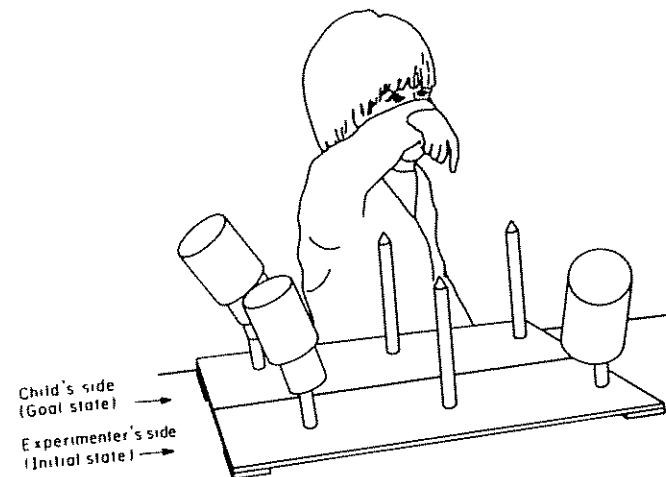


FIG. 7.2 "Monkey cans" arranged for a one-move problem. Initial configuration = state 2; goal configuration = state 1 (see Fig. 7.3).

Externalization of Final Goal. In addition to the current configuration, the goal — or target — configuration is always physically present. We set up the child's cans in a target configuration and the experimenter's cans in the initial configuration. Then the child is asked to tell the experimenter what he (the experimenter) should do in order to get his cans (Es) to look just like the child's. This procedure can be used to elicit multiple-move plans: A child is asked to describe a *sequence* of moves, which the experimenter then executes.

Cover Story. The problem is presented in the context of a story in which the cans are monkeys (large daddy, medium size mommy, small baby), who jump from tree to tree (peg to peg). The child's monkeys are in some good configuration, the experimenter's monkeys are "copycat" monkeys who want to look just like the child's monkeys (more details on the cover story are given later). The cans are redundantly classified by size, color, and family membership to make it easy for the child to refer to them. The subjects find the cover story easy to comprehend and remember, and they readily agree to consider the cans as monkeys.

Formal State Properties

Figure 7.3 shows all possible legal states and all legal moves for these materials. It is called the "state space." No configuration is repeated in the 27 states. The states are indicated by circled numbers, and the can that is moved is indicated by the number on the line connecting adjacent states. The solution to a problem can be represented as a path (a series of states) through the state space. For

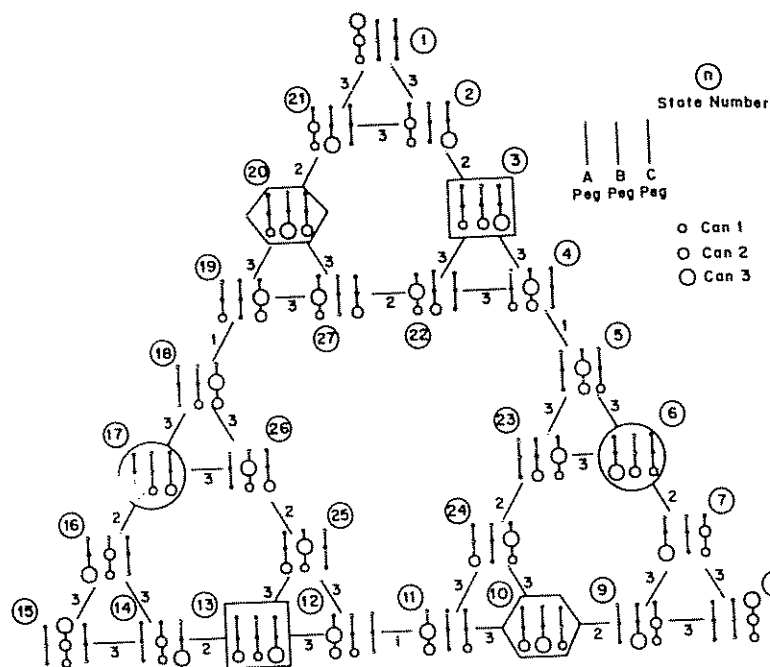


FIG. 7.3 State space for all three-can problems.

example, the minimum solution path for the problem that starts with all three cans on peg A and ends with them on peg C is shown along the right-hand side of the large triangle in Fig. 7.3, moving from state 1 to state 8. The first move involves shifting the largest can (can 3) from peg A to peg C, producing state 2. The next move places can 2 on peg B (state 3), followed by a move of can 3 to peg B (state 4), and so on.

There are no dead ends in this task — any state can be reached from any other state — so that it is possible to consider very many distinct problems (702 to be exact) simply by picking an arbitrary initial and final state. However, there are no two states for which the *minimum* path requires more than seven moves.

Three pairs of special states are indicated by the large squares, circles, and hexagons: these are seven-move problems that begin and end with all pegs occupied. We call these problems “flat-ending” problems, and the “standard” seven-move problems “tower-ending” problems.² As we will see, they have somewhat different properties.

²Tower of Hanoi buffs should note that the “monster problems” of Simon and Hayes (1976) are all five-move flat-ending problems (e.g., (17 to 3, or 13 to 6). An initial state having all the objects on the wrong pegs leads to a shorter solution path than when one of them is already (prematurely) in the right location (e.g., 13 to 3 or 17 to 6).

General Procedure

The general procedure is designed to assess the upper limit (measured by the length of the minimum solution path) of children's ability to solve this problem. The child is introduced to the materials, the rules, and the cover story and presented with a one-move problem (see Fig. 7.2), then a two-move problem, and so on.

SOLUTION STRATEGIES

We can expect children to vary widely in their ability to solve these problems. Indeed, if we look ahead to the most global description of the children's performance, we can see from Table 7.4 that the best subject could reliably solve seven-move problems, whereas the poorest subjects could do no better than two-move problems. To interpret such results, we need to propose hypotheses about the cognitive processes that enable a child to solve problems reliably up to, but not beyond, length n . In this section, we start with a model for perfect performance, then consider — and reject — alternative models for such performance, and finally present a series of “partial” models to account for different degrees of less-than-perfect performance.

Consider the best performers: the 5-year-olds who could reliably solve six- and 7-move problems. When they began to do the task, they knew nothing about the Tower of Hanoi; after 15 or 20 minutes, they were solving our hardest problems with a high degree of confidence. What had they learned? What could they ultimately have acquired as experts on this task?

An Idealized Model

The model to be described is essentially the one first proposed by Simon (1975). He called it the “sophisticated perceptual strategy.” The general procedure is:

1. Compare the current state to the goal state and note all items that are not in their final location.
2. Find the most constrained item (in our case, the smallest can; in the standard form of the problem, the largest disk) that is not yet on its goal peg.
3. Establish the goal of moving that item to its goal peg.
4. Determine the smallest can (if any) that is preventing you from making the desired move.
5. If there is no such can (no culprit), then make the desired move and start all over again (go to step 1).
6. If you can't make the move, then *replace* the current goal with a goal of moving the culprit from its current location to a peg other than the two involved in your current goal.
7. Then return to step 4.

```

SOLVE(C,G)      C = Current state, G = goal state
S1: Find differences between C and G. If none, then done.
S2: n ← (Select smallest can).
S3: New.goal ← (Move can n from X to Y)
      <X = current peg of n, Y = goal peg of n>.
S4: culprit ← TEST (new.goal)
S5: If culprit = nil, then MOVE (nXY); go to S1.
S6: else new.goal ← (Move culprit from X' to Y'); go to S4.
      <X' = current peg of culprit, Y' = other of (X,Y)>

TEST(nXY)
T1: f.list ← See.from(X) <all cans above n on X>
T2: t.list ← See.to(Y) <all cans on Y larger than n>
T3: if f.list = nil & t.list = nil, then culprit ← nil
T4: else culprit ← min(f.list,t.list); exit

```

FIG. 7.4 SOLVE: A set of rules and tests for the "sophisticated perceptual strategy."

Figure 7.4 shows a concise semiformal description of this strategy.³ In addition to the six numbered steps just described, there are four "test" steps that describe the details of the "determine" in step 4. These correspond to a series of perceptual tests used to determine whether there is a culprit blocking the current goal and, if there is more than one, which one should be dealt with first. T1 notes any cans currently above the item whose move is being considered (i.e., on the "from" peg). T2 notes any cans on the current "to" peg that are larger than *n*. T3 tests for whether both of these lists are empty (nil): If they are, then there is no culprit. T4 chooses as culprit the smallest of the obstructors on the combined *f.* ("from") and *t.* ("to") lists.

Figure 7.5 shows the first several steps that SOLVE would take when presented with the seven-move flat-ending problem starting with state 13 (2/1/3) and ending with state 3 (1/2/3).⁴ All the steps listed precede the first move. The full seven moves are shown in Fig. 7.6 (in abbreviated form). The figure shows the series of goals and tests that precede each move. For example, lines 9 to 15 in Fig. 7.6 show that the first goal (line 9) is to get from can 1 from peg B to peg A (G1:1BA). The test detects that can 2 is blocking that goal. Therefore, in line 11 a new goal (G2:2AC) is generated, and so on. Notice that the full listing

³The models in Figs. 7.4 and 7.8 are written as a sequence of FORTRAN- or ALGOL-like steps in which there are several subroutines that are undefined but whose functions are clear from the context. These models are not written as production systems simply because the increased complexity required by such a representation is unwarranted by the level of analysis we are using here (cf. Klahr & Siegler, 1978; Simon, 1975).

⁴The notation (2/1/3) is used to indicate that peg A is occupied by can 2, peg B by 1, and peg C by 3. As another example, state 27 in Fig. 7.3 would be described as 31/-/2.

Change C into G

C: initial state: 2/1/3 (State 13)
G: final state: 1/2/3 (State 3)

S1: What's wrong? 1 not on A, 2 not on B.
S2: What's the smallest misplaced can? 1.
S3: Get 1 from B to A.

S4: Can 1BA be done?
T1: nothing on top of 1
T2: something in the way of 1 on A: 2
T4: 2 is the culprit.
S6: Get 2 out of way of 1BA: Get 2 from A to C.

S4: Can 2AC be done?
T1: nothing on top of 2
T2: something in the way of 2 on C: 3
T4: 3 is culprit.
S6: Get 3 out of the way of 2AC: Get 3 from C to B.

S4: Can 3CB be done?
T1: nothing on top of 3
T2: nothing in the way of 3 on B
T3: No culprit.
S5: Make move: 3 from C to B.

FIG. 7.5 Trace of first few steps of SOLVE on flat-ending problem from state 13 (2/1/3) to state 3 (1/2/3).

of Fig. 7.5 is condensed into lines 9 to 15 in Fig. 7.6. The actual move that is made is shown under "Move" column, and the resulting configuration is shown under "Config." (The last column shows the type of move that this new configuration will require. Move types will be described in the next section.) Figure 7.7 shows SOLVE operating on a standard tower-ending problem (state 15 to state 1).

Simon's (1975) labeling of this strategy as "sophisticated" is apt. The sophistication lies in the strategy's use of the principle that hard problems should be solved before easy ones. More specifically, the most important thing to attend to is the most constrained object.⁵ If it can get to where it is going, the rest will be easier. This principle is utilized in two places: S2 and T4. In S2, the smallest

⁵In Piaget's (1976) study of this puzzle, children were asked about which of the disks traveled the most and whether they traveled more on one problem or another. Although the rationale for this line of inquiry was never described, it may have been related to this same issue. If children are aware of the very different numbers of moves that the disks make during the solution, they may begin to get a rudimentary appreciation of the notion of the relative degree of constraint that the task imposed on the different items.

		Move	Config.	Type
03	13 → 3 SOLVE strategy on flat-ending problem			
04				
05	initial: 2/1/3 Final: 1/2/3			
06				
07			2/1/3	T2
08				
09	G1: 1BA			
10	Test: from (nil), to (2); culprit:2.			
11	G2: 2AC			
12	Test: from (nil), to (3); culprit:3.			
13	G3: 3CB			
14	Test: from (nil), to (nil); culprit:nil.	3CB	2/31/-	T1
15				
16				
17	G1: 1BA			
18	Test: from (3), to (2); smallest:2. culprit:2.			
19	G2: 2AC			
20	Test: from (nil), to (nil), culprit:nil.	2AC	-/31/2	F2
21				
22	G1: 1BA			
23	Test: from (3), to (nil), culprit:3			
24	G4: 3BC			
25	Test: from (nil), to (nil); culprit:nil.	3BC	-/1/32	D3
26				
27				
28	G1: 1BA			
29	Test: from (nil), to (nil), culprit:nil.	1BA	1/-/32	F1
30				
31	G5: 2CB			
32	Test: from (3), to (nil), culprit:3			
33	G6: 3CA			
34	Test: from (nil), to (nil); culprit:nil.	3CA	31/-/2	D2
35				
36				
37	G5: 2CB			
38	Test: from (nil), to (nil), culprit:nil.	2CB	31/2/-	D1
39				
40	G7: 3AC			
41	Test: from (nil), to (nil), culprit:nil.	3AC	1/2/3	goal

FIG. 7.6 Full trace of SOLVE on 13-3 flat-ending problem.

can not yet on its goal peg is selected as the one to attend to. In T4 the smallest (most constrained) of the obstructors of the current goal is chosen as the culprit.

Move Type. Thus far, problems have been characterized by their minimum solution length. It is also instructive to consider the *type* of moves that need to be made. There are three move types, distinguished by the immediate reason for the move.

D moves: Move a can *directly* to its final goal configuration on the target peg.
 F moves: Move a can *from* a peg in order to get to one beneath it.

T moves: Move a can from a peg in order to facilitate moving a different can *to* that peg.

Table 7.1 shows the sequence in which these moves occur along increasingly longer tower-ending solution paths.⁶ The two shortest problems consist entirely of moves *directly* to the goal configuration. Problems of length 3 start with the removal of the largest can *from* on top of the middle-sized can in order to move it (the middle can) on the next move. Four-move problems start with the *direct* move of the smallest can to the goal peg, and so on. Notice that this classification of moves according to why they are taken implies some sort of goal structure on the part of the problem solver.

The basis for this classification is revealed by close examination of Fig. 7.7. If we ask for the proximate reason for any move, the answer is obtained by looking for the immediate supergoal that the accomplishment of the current subgoal will allow. Working from the bottom to the top of Fig. 7.7, we see that D1 and D2 have no supergoals other than the implicit "solve" goal. F1 moves an object that was discovered to be on a *from* peg for the preceding goal (2CA). D3 is another direct move. T moves can 3 again, but this time because it was on the *to* peg of the supergoal. F2 and F3 both involve moving cans from the *from* pegs of the respective supergoals. A similar classification for flat-ending problems is shown in the rightmost column in Fig. 7.7. Notice that for a path length beyond four moves, the flat-ending problems have a different sequence of move types from those of the tower-ending problems.

Alternative Models

The SOLVE model in Fig. 7.4 is very powerful: It will generate the minimum path solution for any three-disk, flat-ending problem and for any *n*-disk, tower-ending problem. Thus, it can be viewed as a possible "cognitive objective" (Greeno, 1976), that is, as the ultimate goal of training someone to perform expertly on this task. What about other functionally equivalent strategies for expert performance? In this section we summarize three quite different strategies, described in detail by Simon (1975), for perfect performance on the Tower of Hanoi. We argue that none of them are as likely to be acquired by our subjects as the sophisticated perceptual strategy represented by SOLVE. Then, in the next section, we describe a series of increasingly powerful partial models that culminate in the SOLVE model.

Goal Recursion. The goal recursion strategy solves the *n*-disk problem by recursively decomposing it into three parts: (1) removal of the *n*-1 disk pyramid from the initial peg to the other peg, (2) moving disk *n* from the initial to the

⁶The numbers following the move type indicate the order of occurrence for that type of move as longer and longer problems are presented. Thus D2 is the second D-move, F3 is the third F-move, and so forth.

TABLE 7.1
Move Types Along Tower-Ending Path

	States		Minimum								
	Initial	Final	Path Length								
			1st	2nd	3rd	4th	5th	6th	7th		
Tower-ending	2	1	D ₁								
	3	1	D ₂	D ₁							
	4	1	F ₁	D ₂	D ₁						
	5	1	D ₃	F ₁	D ₂	D ₁					
	6	1	T	D ₃	F ₁	D ₂	D ₁				
	7	1	F ₂	T	D ₃	F ₁	D ₂	D ₁			
	8	1	F ₃	F ₂	T	D ₃	F ₁	D ₂	D ₁		
	8	1									D ₁

```

03 15 → 1 SOLVE strategy on Tower-ending problem
04
05 Initial: -/321/- Final: 321/-/-
06
07
08
09 G1: 1BA
10 Test: from (33.2), to (nil); smallest:2, culprit:2.
11 G2: 2BC
12 Test: from (3), to (nil), culprit:3.
13 G3: 3BA
14 Test: from (nil), to (nil), culprit:nil. 3BA 3/21/- F2
15
16 G1: 1BA
17 Test: from (2), to (3); smallest:2, culprit:2.
18 G2: 2BC
19 Test: from (nil), to (nil), culprit:nil. 2BC 3/1/2 T
20
21 G1: 1BA
22 Test: from (nil), to (3); culprit:3.
23 G4: 3AC
24 Test: from (nil), to (nil); culprit:nil. 3AC -/1/32 D3
25
26
27 G1: 1BA
28 Test: from (nil), to (nil), culprit:nil. 1BA 1/-/32 F1
29
30 G5: 2CA
31 Test: from (3), to (nil), culprit:3.
32 G6: 3CB
33 Test: from (nil), to (nil), culprit:nil. 3CB 1/3/2 D2
34
35 G5: 2CA
36 Test: from (nil), to (nil); culprit:nil. 2CA 21/3/- D1
37
38
39 G7: 3BA
40 Test: from (nil), to (nil); culprit:nil. 3BA 321/-/- goal
41

```

FIG. 7.7 Full trace of SOLVE on 15-1 tower-ending problem.

final position, and (3) moving the $n-1$ disk pyramid from the other peg to the goal peg. Steps 1 and 3 are accomplished by recursively applying the same strategy. Two considerations make it unlikely that our subjects acquired this strategy. First, they were given only limited exposure to "pyramids" that seem necessary to induce the approach. The only example of recursion they encounter is the seven-move, tower-ending problems. Second, as Simon (1975) notes, to execute this strategy, the full goal stack must be retained in memory. It is both necessary and sufficient for solution: "At each stage, the problem solver can decide what to do next without any reference to the current distribution of the

```

SOLVE.2(C, G)
S1: Find differences (C-G). If none, then done.
S2: n ← (Select smallest)
S3: New.goal ← [nXY]
S4: Move; go to S1.

-----

SOLVE.4(C, G)
S1: Find differences (C-G). If none, then done.
S2: n ← (Select smallest)
S3: New.goal ← [nXY]
S4: culprit ← TEST.4(new.goal)
S5: If culprit = nil, then MOVE[nXY]; go to S1.
S6: else MOVE[culprit, current.of(culprit), empty]; go to S1.

TEST.4(nXY)
T1: l.list ← See.from(X)
T3: culprit ← top(l.list)/nil

-----

SOLVE.5(C, G)
S1: Find differences (C-G). If none, then done.
S2: n ← (Select smallest)
S3: New goal ← [nXY]
S4: culprit ← TEST.5(nXY)
S5: If culprit = nil, then MOVE[nXY]; go to S1.
S6: else MOVE[culprit, current.of(culprit), other.of(X, Y)]; go to S1.

TEST.5(nXY)
T1: l.list ← See.from(X)
T2: l.list ← See.to(Y)
T3: if l.list = nil, then culprit ← top(l.list)/nil,
T4: else culprit ← top(l.list), exit.

```

FIG. 7.8 Partial strategies: SOLVE.2, SOLVE.4, SOLVE.5

disks among the pegs. If only he can retain the unaccomplished part of the goal hierarchy in memory, he can calculate what needs to be done without sight of the puzzle and without holding a visual image of it [p. 270].” Given the highly salient display of both initial *and* final configurations, as well as the severe memory demands of this strategy, it seems unlikely to be acquired by young children.

Notice that the memory demands of the sophisticated perceptual strategy are far smaller. Each new goal needs to be retained only long enough either to make the move directly associated with it or to generate its immediate subgoal. The subgoal *replaces* the immediately prior goal. After a move is made (or imagined), the entire procedure is restarted and the differences between the actual (or imagined) new configuration and the goal (which is physically present) are again determined.

Rote. The move-pattern strategy involves three simple rules that allow one “mindlessly” to solve any tower-ending problem.⁷ (If you really want to impress your friends, you should learn this and then apply it to a seven-disk problem. . . 127 moves without a hitch!) This strategy, like Model IV for the balance scale (Siegler, 1976), is easy to remember and execute, but it is very hard to induce, even for adults. It is quite implausible that our children acquired it in this study.

Another form of rote strategy is based on a memorized list of moves. To solve any specific tower-ending problem, the subject simply cycles through the move sequence he has memorized for that problem. Such a rote strategy could include some degree of generality by being couched in terms of initial, final, and other pegs, which are then instantiated for any particular *n*-disk problem. Another form of rote strategy would consist of a large collection of *S-R* pairs in which *S* is a description of a specific current-final pair of configurations, and *R* is the appropriate move in that situation (e.g., “state 5 to state 8: move 3 to C”). Given the huge number of such specific associations and the limited exposure the subjects get to most of them, this seems to be an unlikely acquisition. Furthermore, it could not explain how subjects who have seen, for example, problems 20-1 and 3-1 immediately solve 6-8 or 13-15.

Partial Strategies

We have rejected as unlikely the major alternative strategies that subjects might acquire as they become expert in this task. Now let us attempt to characterize the different levels of performance in terms of weakened versions of the sophisticated perceptual strategy (SOLVE) just described. Figure 7.8 shows three such partial strategies. In each of the models, steps have been numbered to correspond to the steps in SOLVE (Fig. 7.4). Each model is named according to the length of the problem that precedes the first error the model would make.

SOLVE.2 will solve up to two-move problems, but it will fail on all longer ones except D3. In step 2 it determines which of the differences involves the smallest can, and it establishes (in S3) the goal of moving that can directly to its goal peg. Having set the goal, it then makes the move (S4) without any further perceptual testing or comparisons. When presented with anything other than a D problem, SOLVE.2 makes the wrong move because it immediately attempts to move the smallest can not yet on the goal peg. For anything other

⁷Simon (1975) describes it as follows:

1. On odd-numbered moves, move the smallest disk; 2. On even-numbered moves, move the next-smallest disk that is exposed; 3. Let Peg S be the initial source peg, T the target peg, and O the other peg. Then if the total number of [disks] is odd, the smallest disk is always moved from S to T to O to S, and so on; while if the total number of [disks] is even, the smallest disk is always moved in the opposite cycle: from S to O to T to S, and so on [p. 273].

TABLE 7.2
Move Selected by Partial Strategies for Each Problem Type
Along Tower-Ending Path
goal: 321/-1-

λ	Type	Config. A/B/C	Partial Strategies			
			SOLVE.2	SOLVE.4	SOLVE.5	SOLVE
1	D_1	21/3/-	3BA	3BA	3BA	3BA
2	D_2	1/3/2	2CA	2CA	2CA	2CA
3	F_1	1/-/32	2CA-	3CB	3CB	3CB
4	D_3	-/1/32	1BA	1BA	1BA	1BA
5	T	3/1/2	1BA-	1BA-	3AC	3AC
6	F_2	3/21/-	1BA-	2BC	3AC*	2BC
7	F_3	/321/-	1BA-	3BA, or 3BC*	3BC*	3BA

- illegal

*Off minimum path

than D problems such moves are illegal. SOLVE.2 differs from SOLVE (the model in Fig. 7.4) in three respects: (1) It does not test the feasibility of the move it wants to make, (2) the only goal it sets always includes the ultimate goal peg rather than any temporary, internally generated, subgoal peg, and (3) it never determines the smaller of two obstructors.

SOLVE.4 will solve all problems up to five-move problems. The steps are similar to those in the SOLVE model, with two important exceptions, both contained in S6. When a culprit is detected, S6 does not establish a new goal and then return to S4. Instead, it immediately moves the culprit. Furthermore, S6 does not have the concept of "other". The target peg for the culprit is simply an *empty* peg. Further differences between SOLVE.4 and SOLVE lie in the very weak tests that are used to determine whether a move can be made. These tests determine only whether there is anything on top of the can to be moved. If there is nothing there, the move will be attempted (i.e., there is no culprit, if the f.list is empty).

SOLVE.5 will solve problems up to length 5 correctly and then will begin to make moves that are not on the minimum path. Unlike SOLVE.4, SOLVE.5 now uses the concept of "other" when determining where to move the culprit, but - like SOLVE.4 - it makes the move directly, instead of generating a new goal. SOLVE.5 also lacks the full testing capacity of SOLVE. In particular, TEST.5 does not determine the smallest of the potential obstructors on both the t.list and the f.list. Rather, if the t.list is empty, then the culprit is whatever is sitting at the top of the f.list, if anything. If the t.list is not empty - that is, if there is something larger than the can to be moved on the target peg - then it is assumed to be the obstructor (steps T3 and T4).

Table 7.2 shows the move selected by each of the partial strategies for each of the seven problem types along the tower-ending path from state 15 to state 1.

Consider the F2 configuration in the sixth line of the table. The goal (shown at the top of the table) is to get the three-can stack on peg A. SOLVE.2 would notice that cans 2 and 1 are not on the goal peg. Then it would select can 1 as the can to be moved and would attempt to move can 1 directly to the goal peg, which is, of course, an illegal move. SOLVE.4 would establish, in steps S1, S2, and S3, the goal of moving can 1 from B to A, and then test it. TEST.4 would determine that can 2 on the f.list was the culprit, and steps S5 and S6 would move the culprit, can 2, from where it is, peg B, to an empty peg, in this case peg C. This turns out to be - serendipitously - the correct move. SOLVE.5 would also enter the test phase with the goal of moving can 1 to A. However, T3 would fail, since the t.list - the list of cans on the "to" peg that are larger than the can to be moved - would contain can 3. Steps 5 and 6 in SOLVE.5 would then move can 3 to the "other" peg, that is, move 3 from A to C, which, although legal, is not on the minimum path. Of course, SOLVE would correctly decide to move 2 from B to C. (The details are shown in lines 16 to 19 of Fig. 7.7.)

The pass/fail patterns in Table 7.2 are predicted not only for the particular configurations shown but also for any configurations of the specified type. Thus, they suggest an empirical approach similar to that used by Siegler (this volume) for determining which, if any, of the strategies a child is using. However, such a procedure assumes that the strategy being used remains stable during the assessment. Although the stability assumption is reasonable in cases where the assessment yields no feedback for the subject (as in a typical pretest condition), it is untenable in the situation that our subjects faced. For each item, they were required to produce a solution, and they were well aware of how successful they had been. Furthermore, in this initial study, the problem sequence was designed to assess only their first point of failure. Thus, although the partial models appear to have a potential for precise diagnosis, in this chapter we utilize only the predictions they make about the longest problem reliably solved.

A STUDY OF CHILDREN'S PERFORMANCE

The study was run in two phases, with the second to some extent being contingent on the outcome of the first. The two procedures are described separately, and then their results are combined in the analysis. The main difference between the phases lies in the age groups used and the amount of intervention by the experimenter.

Subjects

Thirty children attending the Carnegie-Mellon University Children's School participated. There were 10 children in each of three age groups "3s" (mean 3:10, range 3:8 to 4:4), "4s" (mean 4:5, range 4:1 to 4:9), and "5s" (mean

5:9, range 5:2 to 6:3). The sex ratio was approximately 50/50 in each age range. The children came from predominantly – but not exclusively – upper-middle-class professional families.

Phase I: Purpose and Procedure

There were three goals for the first part of the study, which used the 4-year-old group: (1) Explore the basic ability of uninstructed 4-year-old children to solve problems of various lengths; (2) explore the extent to which they could describe a multiple move sequence; and (3) explore the effectiveness of some rudimentary instruction, including a graduated sequence of problems and a few simple hints about goals and subgoals.

The child was familiarized with the materials shown in Fig. 7.2. Then the rules and objectives were described in the context of a cover story. The cover story went something like this:

Once upon a time there was a blue river [point to space between rows of pegs]. On your side of the river there were three brown trees. Can you count your trees? On my side there were also three brown trees. On your side there lived three monkeys: a big yellow daddy [put yellow can on a peg], a medium-sized blue mommy, and a little red baby. The monkeys like to jump from tree to tree [according to the rules]; they live on your side of the river. On my side there are also three: a daddy, [etc.] Mine are copycat monkeys. They want to look just like yours, right across the river from yours. Yours are all stacked up like so [state 1], mine are like so [state 2 or 21]. Mine are very unhappy; can you tell me what to do so mine can look like yours?" [The actual script is, of course, more elaborate.]

The problem sequence was generated by choosing increasingly longer problems from alternating sides of the two tower-ending paths terminating in state 1 (Fig. 7.3), for example, 21-1, 3-1, 19-1, 5-1, and so on. If the child suggested an illegal move, the experimenter would point out the illegality. If the child had no suggestion for a move, the experimenter would make the correct first move and then ask the child to continue. If the child was successful at completing the problem in the minimum number of moves, the experimenter would give him either another problem of the same length from the other side of the state space triangle or one that was one move longer. The idea behind all of this was to get an estimate of the upper performance level of the child without either having the child give up because of too many failures or generating lucky solutions.

As the child appeared to reach his upper limit, the experimenter would begin to give a systematic series of hints, attempting to move the child through the steps of the SOLVE strategy described earlier.

For the first few problems, the children were asked to state not just the next move but the next *several* moves. Children varied widely in their ability to do

TABLE 7.3
Pairs of Initial → Final States for Problems of Different
Path Length

Path Length	Problem Order	Tower-ending	Flat-ending
3	a	19 → 1	19 → 3
	b	4 → 1	23 → 3
	c	18 → 15	22 → 6
	d	5 → 8	11 → 6
4	a	5 → 1	18 → 3
	b	18 → 1	24 → 3
	c	19 → 15	27 → 6
	d	4 → 8	12 → 6
5	a	17 → 1	26 → 3
	b	6 → 1	11 → 3
	c	20 → 15	19 → 6
	d	3 → 8	25 → 6
6	a	7 → 1	25 → 3
	b	16 → 1	12 → 3
	c	21 → 15	18 → 6
	d	2 → 8	26 → 6
7	a	15 → 1	14 → 3
	b	8 → 1	16 → 6
	c	1 → 15	13 → 3
	d	1 → 8	17 → 6

this, and the experimenter tried to adapt to these variations. For many children, the attempt to elicit multiple-move plans was given up after the first few problems.

Phase II: Purpose and Procedure

The three objectives of Phase II were as follows:

1. Further investigation of the effects of types of moves. This required varying the goal configuration so that the possibility of specific stimulus configurations being associated with specific moves could be ruled out. This was accomplished in two ways. First, both tower-ending and flat-ending problems were presented. Second, within a problem type, the goal configuration was systematically changed.
2. Elimination of the confounding of the two types of instruction. In this phase, no guidance was given when children reached their peak performance.

TABLE 7.4
Number of children in Each Age Group Who Could Reliably Solve Tower-ending
Problems Up to But Not Beyond Given Length

Nominal	Age		Problem Length (λ)							λ	
	Mean	Range	n	1	2	3	4	5	6		7
3	3:10	3:8 - 4:4	10	0	5	1	4	0	0	0	2.9 (0.99)
4	4:5	4:1 - 4:9	10	0	1	1	5	1	2	0	4.2 (1.2)
5	5:9	5:2 - 6:3	10	0	0	0	0	5	4	1	5.6 (0.7)
			30	0	6	2	9	6	6	1	
			Cumulative ^a	30	30	24	22	13	7	1	
			Nontransition probability	0.00	0.20	0.08	0.40	0.46	0.86		

^aTotal number (all groups) solving problems up to and including given problem length

The only "instruction" consisted of presenting the increasingly longer problem sequence and making the correct first move when subjects did not know what to do.

3. Beginning to measure age-related differences on this task. The 3- and 5-year-old children described at the beginning of this section served as subjects in Phase II. The apparatus, cover story, instructions, and so on were identical except, as just mentioned, no hints were given when subjects began to falter.

Table 7.4 lists the initial and final states for the problems used in this study. (Reference to a problem as having " n -moves" means that the minimum path has n moves). Of course, one could solve an n -move problem in *more* than n moves. All the T-T problems end with states 1, 8, or 15 and start somewhere on the outer contours of the exterior triangle in Fig. 7.3. All minimum paths for T-T problems are also along these contours. All the flat-ending problems end in states 3 or 6 and start in some state on the interior irregular hexagon in Fig. 7.3 (except for the seven-move, flat-ending problems). Notice that along the minimum path for all flat-ending problems there are no flat configurations other than the final one (again, except for the seven-move, flat-ending problems).

The problem sequence was generated by selecting initial-final pairs as follows. The first two problems were 21-1 (one-move) and 3-1 (two-move). (These were always solved correctly.) Then a series of tower-ending problems was selected from the list in Table 7.4. For a given problem length, n , problem a was presented first. If it was correctly and confidently solved, then n was incremented by 1, and problem a of the next length was presented. If there was hesitation or apparent uncertainty, problem b of the same length was presented. If there were any errors, the problem was reinitialized and the first move was made by the experimenter (this would convert the a problem of length n into the b problem of length $n - 1$). If two of the three remaining problems were correctly solved, then n was increased again. Otherwise, the T-T sequence was terminated, and a shift to flat-ending problems was made, starting with length $n-2$, where n was the current level for the tower-ending series. This procedure was designed to present a series of longer and longer problems ending in state 1 until the subject reached his upper limit. Then problems ending in states 15 and 8 were presented to assess one level of generality of what had been learned. Following this, the flat-ending series was presented to test generalization still further.

Results

Path Length. At any point in the solution of a problem, the child can suggest either the correct move or several kinds of "incorrect" moves. These include legal moves not on the minimum path, illegal moves, and "don't knows."

TABLE 7.5
Number of Children Reliably Solving Tower-ending and Flat-ending
Problems of Each Length^a

		Length of Tower-ending						
		2	3	4	5	6	7	
Length of Flat-ending	1	1	-	-	-	-	-	1
	2	2	-	2	-	-	-	4
	3	-	1	1	1	-	-	3
	4	-	-	-	4	3	-	7
	5	-	-	-	-	1	-	1
	6	-	-	-	-	-	1	1
		3	1	3	5	4	1	

^aThree-year-olds ($n = 7$), 5-year-olds ($n = 10$).

A global measure of the child's ability is the length of the longest problem (or subproblem) for which he reliably stays on the minimum path.

The distribution of subjects reliably solving problems of a given length is shown in Table 7.5. The age differences are clear and striking. The regularity of the age effect indicates that the semiclinical procedures used in Phase I did not seriously distort the assessment of maximum performance levels, although it may have increased the spread of the 4-year-old group somewhat.

Consider first the 4-year-olds from Phase I. None of them could solve the seven-move, tower-ending problem reliably. However, two of them could do the six-move problem, that is, they could solve the seven-move problem if the first move had already been made. One child could do no better than the two-move problem, and most of the children could reliably do up to four-move problems before they began to err.

Five-year-olds were just about evenly divided between five- and six-move problems, and one of them could solve seven-move problems. The 3-year-olds fell into two major groups: those who could not get beyond two moves and those who could do four-move problems.

Recall that in the tower-ending series, subjects received problems ending in different goal states to ensure that they had not simply learned moves from specific stimulus configurations. Thus, we are fairly confident that a subject classified as an n -move child could solve problems starting n moves away from any tower. Flat-ending problem performance further assesses the generality of what subjects know about the problem. Table 7.5 is a cross classification of three- and five-year-old children according to their maximum performance on tower-ending and flat-ending problems. (Only seven of the three-year-olds are included here. The others were never run on flat-ending problems). The data suggest that flat-ending problems are uniformly harder. For example, of the four

children who could solve up to six-move, tower-ending problems, only one could solve the five-move, flat-ending problem, and three could solve only four-move, flat-ending problems. No child could solve a flat-ending problem beyond his tower-ending level, and most of them dropped down two levels. Table 7.5 is also very regular in that the 10 entries in the lower right-hand quadrant are all 5-year-olds, the others are all 3 years of age. These results are consistent with the partial models in a weak sense. It can be demonstrated that for problems beyond length 2, the partial models make nonoptimal or illegal moves on flat problems before they make any errors on tower problems of the same length.

The decline in performance as path length increases is very irregular, as shown in the last line in Table 7.5, indicating the conditional probability that a subject who can solve an n -move problem will fail an $n + 1$ -move problem. Of the 30 children who could solve two-move problems, six (20%) could go no further; of the 24 who solved three-move problems, only two (8%) could not solve four-move problems; and so on. For the total group of subjects, it is clear that very few "peak" at three-move problems: They can either do no better than two-move problems or they go on to the longer ones.

These results motivated the set of partial models described earlier. There is no SOLVE.1 because every subject could solve at least two-move problems. The other SOLVE models were proposed because of the substantial frequency with which subjects at levels 4 and 5 could go no further. Although not shown, a SOLVE.6 model can be derived from SOLVE.5 by removing the capacity to generate a new goal, that is, by replacing S6 in SOLVE.5 with S6 from SOLVE.5.

One measure of the relative difficulty of different move types is the relative frequency with which the correct move is made for problems starting with that type of move.⁸

Figure 7.9 shows the mean relative frequency with which 4-year-olds make the correct first move on problems starting with different types of moves. Performance on D1 and D2 is perfect on first encounter: For D1, there is little that the child could do wrong, and for D2, the major problem is to determine the correct order in which to move the two cans to the goal peg.

F1 requires the removal of the largest can from the medium one. The most frequent error here is an attempt to move the medium can directly, without first removing its obstructor. Another frequent error is to attempt to move the medium one while the large one is held aloft or while the large one is temporarily placed in the "river." The incorrect moves here are almost always such illegal

⁸In a sequence of moves, the heuristic of never moving the same can twice in a row gives a 0.5 probability that the correct move will be made. However, for initial moves, the chance level is only 0.33 (except for the first move in the seven-move problems, where it is 0.5). Even this overestimates the chance probability of a correct move, since subjects may suggest illegal moves as well.

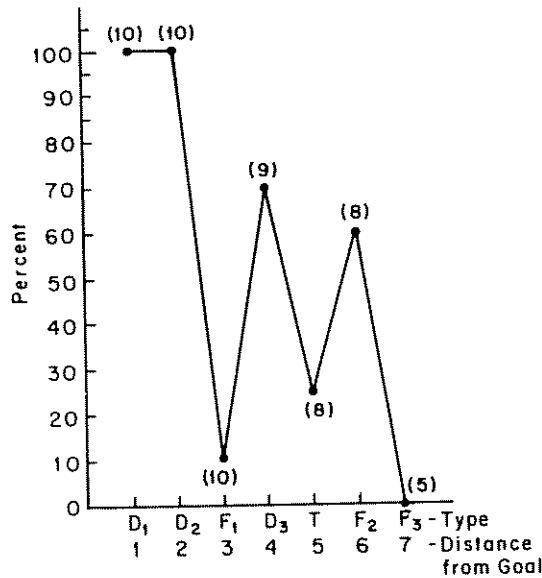


FIG. 7.9 Mean relative frequency of correct first move for problems starting with a given move type (number of subjects).

suggestions; rarely are they either of the two legal moves off the minimum path. In our procedure, the F1 items represent the child's first "real" problem; as Figure 7.9 shows, few of the children know what to do at this point.

Although D3 is four steps from the goal, it is a move that would *a priori* seem obvious, but 30% of the children err on their first encounter with it. Since all the models predict a correct move here, this result poses another challenge to the proposed partial models.

Move T is unique. It is the only one requiring that a can be moved *away* from its ultimate goal peg, and it has a low initial score. SOLVE.5 is the first model with sufficient power to solve T problems.

This analysis reemphasizes what the models are intended to convey: The relationship between initial and final states is not simply one of distance in the state space. Instead, the *type* of move — implicitly characterized by the mental operations that generate it — appears to be of central importance.

Learning. The distinctiveness of the move types is further revealed by an analysis of the rate at which they are learned. One rough measure of the rate of learning is a plot of frequency of correct moves as a function of the occurrence of that type of move. The 4-year-olds' "learning curves" are shown in Fig. 7.10 for four move types. Figure 7.10 shows, for example, that the first time move type F1 is encountered, it is never made correctly. For 50% of the second

occurrences of F1, the correct move is made, and by the third time any particular subject has encountered an F1 move, the probability is about 0.9 that the correct move will be made.

The quotation marks around "learning curves" are a reminder of the rather complex data base that underlies them. Recall that the first occurrence of a move type is also a problem that *starts* with that move. Subsequent occurrences are almost always reached *en passant* from longer problems. However, this is not invariably the case, because the experimenter would, for example, occasionally drop back to a 3 or 4-move problem if the subject had erred on a longer problem when only three or four moves from the goal. Furthermore, as subjects reached their peak performance, they dropped out of the data base completely, so that the later occurrences of any move type include only the higher performing subjects. For example the 9 points for the F1 curve are based on the following ratios of correct moves to number of instances: 0/10, 5/10, 8/9, 8/8, 7/7, 5/6, 3/3, 2/2, 2/2. For F2 the ratios are 5/8, 3/5, 2/3. As a result of these complications, the curves in Fig. 7.10 overstate the rate at which the moves are learned and understate the differences in the types of moves.

What could subjects be learning? First, let us consider what they are *not* learning. The data lend no support to the notion that what subjects learn is a list of moves. That is, having learned that from state 2 they can get to the goal (state 1), and from state 3 to state 2, subjects do not, upon first encountering state 4 (requiring an F1 move), search for a move that produces a familiar state (3) that is known to lead to a solution. If this were the case, then the data for first-move accuracy (Fig. 7.9) and acquisition (Fig. 7.10) would be much more

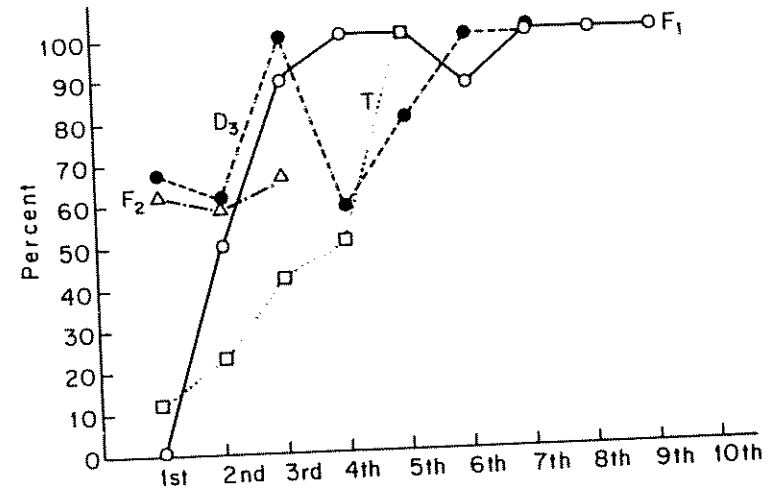


FIG. 7.10 Frequency of correct moves for each move type on nth occurrence (4-year-olds).

systematic than is actually the case. For example, even though D2, when it is physically present, always produces the correct move, when it is one of several possible states that could be reached from F1, it is not immediately recognized as being a desirable subgoal along the solution path. Figure 7.10 suggests that the F1 and D3 moves are acquired quickly relative to the T, F2, and F3 moves. Success on these first two move types corresponds directly to the transition between SOLVE.2 and SOLVE.4. The additions to the SOLVE.2 strategy are a simple test before attempting a move (S4, and TEST.4 in SOLVE.4) and a direct action contingent on the outcome of that test (S5 and S6 in SOLVE.4). The transition from SOLVE.4 to more advanced strategies requires an elaboration of the test for legal moves, the introduction of the concept of the "other" peg, and for SOLVE (i.e., for F3 problems), the generation of a new goal rather than a direct move. As suggested by Fig. 7.10, these acquisitions and their organization posed a formidable challenge to our subjects.

Mixed Strategies. Subjects who manage to get beyond the three-move, tower-ending problems almost invariably run off the last three moves very rapidly and smoothly. Even with the experimenter moving the cans, this sequence, which always involves moving a two-can stack to the goal peg, takes no more than a few seconds to execute. Recall that this is the sequence starting with F1, the move that is never correctly made on first occurrence. From first occurrence to last, there is a reduction by a factor of at least five in the time it takes to run off the three moves.

What seems to be happening is a shift from a simple, direct strategy (SOLVE.2), to one that has additional steps and tests (SOLVE.3), to the further development of a specialized local procedure. As SOLVE.3 first develops, the child appears to consider the differences between initial and final states and to proceed in a means-ends fashion to reduce those differences. After several such experiences, he acquires not just the single correct move but the entire sequence. For example, in the case of F1, what appears to be acquired is the three-move sequence for moving a two-can stack to the goal peg. A possible form for the "subroutine" is

```
Goal: [32 → Goal.peg]
Move [3 → empty.peg]
Move [2 → Goal.peg]
Move [3 → Goal.peg]
```

This representation is "rote" in that it generates a move sequence without any further reference to the stimulus configuration, once it is evoked by the need to move the 32 stack. It is "general" to the extent that the peg containing the stack and the goal.peg are variables to be instantiated by the particular evoking circumstance. However, the goal.peg must be the peg in the final goal and not

19300	P5	E 3/1/2	
19400		S 321/-/-	
19500	5:44	S: Like that, like that.	
		E: Like that, and they all want to be over here (A).	
		S: Well, that is pretty hard, but I can do it.	
19800	5:49	E: O.K.	
19900		S: Oh. Take... I'm thinking.	
20000	5:57	E: What are you thinking about?	
20200	6:10	S: How we should do this.	
20300	6:12	E: Tell me what you're thinking?	
20400		S: About how to do this.	
20500	6:15	E: O.K. What do you think? How should we do it?	
20600		S: I don't know yet.	
20800	6:19	E: Take the yellow (3) off and put the yellow one on here (C),	
20900	6:38	and then take the red (1) and put it on there (A), and take the...	
21000		and then put the yellow one and put it on here (B), and then	
21100		put the blue one (2) on the red, and then put the yellow one on	
21200		the blue one.	
21300			
21400	6:54	E: Let's try that.	
21500		What's the first thing I should do?	
21600		You said...	
21700	7:01	S: Take this one (3) and put it on there (C)...etc.	

23400	P6	E 3/-/21	
23500		S 321/-/-	
23600			
23700	7:26	S: Oh. that.	
23800	7:31	O.K. That's easy.	
23900		Just take the yellow one and put it on there (B).	
24000	7:37	Take the (pointing to 2(C)...and take...and take, take the b...	
24100	7:41	No, take the blue one, put it on there (B), and then, then,	
24200		take the yellow and put it on the blue (points toward C, then	
24300		to B), and then take the red one and put it on here (A)).	
24400	7:51	And then take the blue one and...No, and then...and then put	
24500		the yellow one here (C), and then put the blue one on the	
24600		red one, and then put the yellow one on the blue one.	
24700	8:04	E: O.K. What's first?	
24800	8:07	S: First, take this (3) and put it on there (B).	
24900		E: (3 → B)	
25000	8:13	S: No, I was wrong. Forget it.	
25100		(E puts 3 back on A).	
25200	8:15	First take the blue one and put it on there (B).	
25300		E: (2 → B)	
25400		S: Now take the yellow one and put it on there (B)...etc.	

FIG. 7.11 Protocol from subject aged 4:11, showing five-move (a) and six-move (b) plans.

some intermediate goal. This limited generally keeps this routine from being evoked at the start of a seven-move problem. Even if the goal of moving the two top cans off the smallest was generated at this point, the routine just described would not be evoked because the target for this first placement of the two-can stack is not the goal peg. A further limitation of this routine is its ineffectiveness on flat-ending problems.

All the partial strategies described earlier (except for SOLVE.2) are likely to have something like this routine in addition to the steps shown in Fig. 7.5. When it is evoked, it takes over, running off the last few moves. However, since it is of no use on flat-ending problems, the strategies listed would maintain control to the very end.

Planning. At times, the experimenter asked the child for not only the next move but the next several moves. Few children verbalized multiple-move sequences beyond a few moves. Repeated requests to do so appeared to confuse them and had to be terminated. However, those who could and would verbalize showed a remarkable ability to describe a series of imagined future states and actions on those states.

Perhaps the most impressive plans are shown in Fig. 7.11. They came from a boy, aged 4:11. At the top of Fig. 7.11, the complete five-move plan is verbalized after about 60 sec of study (lines 209–214). The next problem for the same child has a six-move minimum path. The child starts verbalizing a move sequence after just a few seconds, (239–240), corrects himself, and then smoothly rattles off a six-move plan (241–246). Notice that when he starts to make actual moves, he makes the same initial error and recovery (248–252) that was in his verbalizations.

This protocol (as well as several others not included here) is in striking contrast to Piaget's results and interpretations of his investigations of the same task (Piaget, 1976). He reports that before the age of 7 or 8 years children did not plan, even with the two-disk problem. He concludes that they cannot "combine inversion of the order" (in our context, putting the baby under the mommy, while the mommy is moved first) with a "sort of transitivity" (using an intermediate peg to hold a can temporarily). There is, Piaget says, "a systematic primacy of the trial-and-error procedure over any attempt at deduction, and no cognizance of any correct solution arrived at by chance [p. 291]." I find it hard to reconcile this interpretation with these protocols.

Summary of Results

The following picture is emerging: (1) Path length is a factor in problem difficulty, but it is confounded with move type, which seems to be much more important. (2) Path type is important, but we cannot conclude that flat-ending problems are intrinsically more difficult than tower-ending problems because

they are always given after tower-ending problems. Rather, we can conclude that whatever is acquired on tower-ending problems is less effective on flat-ending problems. (I will hazard a guess, though, that even with a different training sequence, flat-ending problems will turn out to be harder. The reason for this prediction is given later.) (3) Age effects are clear, significant, and monotonic. There are no U-shaped performance curves. (4) Planning, or at least verbalization of move sequences, is difficult to elicit. The rare occurrences are, however, quite impressive. Notable in all the protocols is the *absence* of a seriously incorrect plan. Subjects either can say what they are going to do correctly or they say nothing at all. (5) Learning rates are very sensitive to move type.

Goal Ordering

Recall the earlier comments about the locus of "sophistication" in the SOLVE model. Built into every model, as the second step, is the knowledge that the smallest cans must be attended to first. No child, even the 3-year-olds, ever attempted to invert the order of moves on two-move problems. This ordering takes place every time a new situation is assessed (Step 2) and, for the six- and seven-move problems, when the smallest obstructor is sought.

One explanation for the flat-ending performance being poorer than tower-ending performance may be that the latter provides a highly salient external representation for the ordering of the goals, whereas the former does not. A target display such as (321/-/-) provides very strong clues that the last can to be placed is 3, the second from last 2, and the first 1. However, a target such as (1/2/3) provides no such compelling reminders that small things must be taken care of before large things. In fact, for all the seven-move, flat-ending problems, there are *two* minimum paths. One of them achieves its goals in the "proper" order, and the other one does not. (See, for example, 13 to 3 via 24 in the state diagram, Fig. 7.3.) The more likely that the correct goal ordering is not maintained, the longer the solution path will be, because incorrect goals are generated during solution. This is exactly what happened: The average solution path length is longer for all flat-ending problems than for corresponding tower-ending problems. There appears to be much more meandering around the state space with flat-ending problems. For tower-ending problems, the goal display may serve as a physical manifestation of the notion of a "goal stack."

CONCLUDING COMMENTS

What develops? That is the theme of this text, and I suppose I should take a stab at an answer. I must remind you that at this point in the research program only the roughest outlines of a picture are emerging, and much remains to be done

even on this one task, not to mention others. In proposing an answer I go beyond the limits of the study described here and try to characterize the ways in which the question might be answered. There seem to be three kinds of answers. I call them the *empirical*, the *characteristic*, and the *procedural*.

Empirical descriptions consist of performance measures and their changes over time. Typically, we see monotonic improvements with age, but often a careful observational paradigm reveals dips and peaks on specific tasks. It seems to me that improvements in both the analysis and the measurement procedures used in cognitive development research have substantially enhanced the precision and interest of answers to what develops simply by pointing to the data. For example, in Siegler's (1976) balance scale task we see dramatically different paths in the pass/fail patterns for different problem types. For the task studied here, Table 7.4 is an example of an empirical description of what develops.

A second way to talk about what develops is to propose *global characteristics of the child*. The child is wholistic or analytic, he is rule-governed or not, he is familiar with specific stimulus materials, he is systematic, he is egocentric or metamemory. For the Tower of Hanoi problem, the characteristic description of what develops includes terms like orderliness, planfulness, undistractability, and focus. Sequential ordering is crucial to problem solutions; furthermore, the child must focus on the newly generated subgoal and not be misled by the still unsatisfied supergoal. (In the model proposed, such supergoals are supposed to be obliterated by new goals, but it is difficult to forget on command.) The child must be able to decompose the desired end state into a series of attainable, temporally ordered, intermediate states, and he must then reintegrate the parts into a whole.

Finally, we can formulate *procedural descriptions* of what develops. The Tower of Hanoi models provide examples. At the present stage of this project, the proposed models are but the roughest approximations to what children know when they exhibit a particular performance level on this task. Younger children's strategies contain no tests: They are very direct in their attempt to get to the goal. Older children make tests before they move, but they still move directly, rather than by generating new goals that are further tested. Only the most advanced children have the ability both to generate subgoals and to utilize the concept of the "other" peg.

But remember, there is more to the developmental story – even to the procedural account – than just performance models. The full story of what develops must account for the psychological processes that enable a child to listen to the task instructions, assimilate them to his existing general problem-solving processes, and produce something approximating the performance models we have previously described. There are several components to that sequence, and a child who can do only two-move problems may differ from one who does seven-move problems with respect to any of them. His general problem-solving abilities may be less, his assimilation capacity may be inadequate,

or his information-processing system may lack the capacity to run the task-specific model (cf. Klahr, 1976; Klahr & Wallace, 1976).

There seem to be ways of isolating these components. For example, in a study with adult subjects Neves (1977) is teaching different Tower of Hanoi strategies with direct instruction to get precise measures of the demands they make on the information-processing system. This procedure eliminates the effects of the general problem-solving capacity and the assimilation procedure. In fact, subjects are not even told what the problem is, they are told only how to go about making a move. We can shed some light on the child's general capacity by extending the sort of analysis described in this chapter to a range of other problems, and such studies are in progress in my own lab. Finally, we can study the assimilation problem by adding to the precision of the measures. A cleaner procedure for getting at latencies and eye movements (see also Neves, 1977) would enable us to trace the information of the kind of subroutines – or procedural chunks – described earlier.

It is probably clear that, of the three forms of answer to the question of what develops, my preference lies with the procedural. It seems to me that, no matter what the domain we are studying, we ultimately must move through characteristic and empirical accounts to procedural descriptions of what develops. Then we can all get to work on the really interesting question: *How?*

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8

Counting in the Preschooler:
What Does and
Does Not Develop

Rochel Gelman
University of Pennsylvania

For some years now, I have contended that preschool children can and do count to represent the numerical value of a set of objects or pictures. I do not mean that the young child who is able to rattle off the number words is necessarily able to count. He may or may not be able to do so. It all depends on what else he can do with the list of words he rattles off. And as we shall see, the young child who is unable to rattle off the conventional words in the conventional order may nevertheless be able to count. In short, I do not rest my claim that young children can count on their ability to recite the conventional number words. If not this, then what? To what kind of evidence can I possibly be appealing? To answer this question, it is necessary to consider what is involved in counting. Thus, I begin my discussion with a summary of a counting model on which my husband and I have been working (Gelman & Gallistel, 1977). Next, I present data on the extent to which young children's "counts" are governed by the counting principles outlined in the model.¹ Finally, I address the questions of what does and what does not develop.

THE COUNTING MODEL

What does it mean to say that a young child counts to represent number? Inspection of the various count sequences that I have recorded led me to the view that the young child's ability to count is governed by several principles and

¹The word *count* is in quotation marks to reflect the fact that we have yet to define the ability to count.